MATH 306 Workshop

1. Define what it means for a list $v_1, v_2, \dots, v_n$ of vectors in V to be **linearly independent**.

2. Define what it means for a list $v_1, v_2, \dots, v_n$ to be a **basis** for V.

3. Define the **dimension** of a finite-dimensional vector space.

4. Define a **linear map** from V to W.

5. State **Fundamental Theorem of Linear Maps**.

6. Prove or disprove: the list $(1,1,1), (1,1,0), (1,0,0)$ is a basis for $\mathbb{R}^3$.

7. True or False (If it is false, give a counterexample or explain):

    a. $\mathbb{R}^n$ has a unique additive inverse and a unique multiplicative inverse.

    b. In $\mathbb{R}^5$, $5 + (1,2,3,4,5) = (6,7,8,9,10)$.

c. $\{(x_1, x_2, x_1 + x_2) : x_1, x_2 \in \mathbb{F}\}$ *is a subspace of* $\mathbb{F}^3$

d. $\{(x_1, x_2, x_1 + 5) : x_1, x_2 \in \mathbb{F}\}$ *is a subspace of* $\mathbb{F}^3$

e. If U and V are two subspaces of V, then $U \cap W$ is also a subspace of V

f. Every linearly independent list of vectors in V with length dim V is a basis of V

8. Recall that $\mathcal{P}(\mathbb{F})$ is the set of all polynomials with coefficients in $\mathbb{F}$.

   a. What does $\mathcal{P}_3(\mathbb{F})$ stands for?

   b. Give an example of a basis of $\mathcal{P}_3(\mathbb{F})$.

9. Assume U and W are subspaces of $\mathbb{F}^{10}$ with $dim(U) = 8$ and $dim(W) = 6$. What is the minimum possible value of $dim(U \cap W)$?

10. Explain why there does not exist a list of six polynomials that is linearly independent in $\mathcal{P}_4(\mathbb{F})$.

11. Let $V = \mathbb{F}^n$. Let U be the subspace of V consisting of all vectors $y = (y_1, \ldots, y_n) \in \mathbb{F}^n$ such that $y_1 + \ldots + y_n = 0$.

    a. Find a basis for U and prove that your answer is a basis.

    b. What is the dimension of U?

12. Prove that if U_1 and U_2 are subspaces of V, then $U_1 + U_2$ is a subspace of V.

13. Let $U = \{p \in \mathcal{P}_4(\mathbb{F}) : p'(1) = 0\}$.

a. Prove that U is a subspace of $\mathcal{P}_4(\mathbb{F})$.

b. Find a basis of U. What is the dimension of U?

14. Let U be the subspace of $\mathbb{F}^5$ defined by:

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5 : x_2 = x_1,\ x_3 = 2x_4 - 3x_5\}$$

(a) Find a basis for U and prove your answer is a basis.

(b) Let W be the subspace of $\mathbb{F}^5$ defined by:

$$W = span\{(1,0,0,0,0), (0,0,1,0,0)\}$$

Show that $\mathbb{F}^5 = U \oplus W$

15. Let U be the subspace of $\mathbb{C}^5$ defined by:

$$U = \{(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 : 6z_1 = z_2,\ z_3 + 2z_4 + 3z_5 = 0\}$$

(a) Find a basis for U.

(b) Extend the basis in part (a) to a basis of $\mathbb{C}^5$.

(c) Find a subspace W of $\mathbb{C}^5$ such that $\mathbb{C}^{\mathbf{5}} = U \oplus W$

16. Given the integration map $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ by $T(p) = \int_0^1 p(x)dx$

Prove or disprove it is a linear map.

17. Given the map $T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$ by $T(x, y, z) = (2x + 5y, 3z)$

1) Prove or disprove it is a linear map.

2) If it is a linear map, find a basis for null T.

3) What is the dimension of Range T ?